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FORCE AT A POINT IN THE INTERIOR OF A LAYERED ELASTIC HALF SPACE

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Abstract—This study formulates, by the technique of integral transforms, the solution of a layered half space subjected to a concentrated force which may act either vertically or horizontally in the interior of the system. Accurate approximations of the reciprocals of the common denominators in the solution integrals are suggested in such a way that the latter are in standard closed forms and can be identified by two parts. The first part is the singular part of Mindlin's solution which is singular at the point of application of the force, and the second is non-singular. The solutions for plane problems are also obtained in closed forms by performing appropriate integrations of the solutions for the corresponding three-dimensional cases.

1. INTRODUCTION

The static behavior of a multi-layer elastic half space has been a subject of interest since it represents a closer approximation to many actual conditions of real soil in natural deposits, which are layered in character. Adequate investigations for cases due to a force applied in the interior of such a system should give more reliable informations for the analysis and design of foundation systems such as the anchor of guy wire and suspension cable, etc.

General solution for two- and three-layer elastic half space in integral forms was first given by Burmister[1, 2]. Subsequently, with the approximation of the reciprocal of the common denominator involved in the integrals, the same author presented a complete numerical solution for displacements and stresses in a layer with an underlying rigid base, and loaded by a vertical concentrated force on the free surface[3]. By integration of Burmister's solution, Poulos[4] presented the solution for any general shape of loaded area. A two- or three-layer system loaded by a circular footing was studied by Acum and Fox[5].

The use of the technique of Hankel transforms in the analysis of axisymmetric problems of an elastic half space subjected to indentation was presented by Harding and Sneddon[6], and it was extended by Muki[7] to asymmetric problems. Westmann[8] generalized Muki's formulation and obtained the solution for a two-layer system subjected to a surface shear. Recently, Chen[9] presented a general formulation in the form of Fourier integrals in Cartesian coordinates of a three-layer system, which is applicable for vertical loading over a region of arbitrary shape on the surface. To improve the convergence of the numerical integration, the solution integrals of these systems were subtracted by the ones for a homogeneous half space and the results were evaluated by the Gaussian quadrature method. For a system consisting of many layers, a matrix method to reduce the computational work involved

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was proposed by Kuo[10] and Thrower[11]. Another general formulation by the technique of Fourier transforms were derived by Lemcoe[12] for plane strain problem of a two-layer system. The solution integrals were evaluated by numerical integration using Simpson's rule. All of these works are limited to cases of forces acting on the surface of the half space.

Force at a point in the interior of a homogeneous elastic half space was first studied by Mindlin[13]. The solution was obtained from Kelvin's solution by the method of synthesis and superposition. This classical method requires ingenious guesses of the proper potential functions. Mindlin reconsidered the same problem later[14] and solved it directly by means of Papkovitch potential functions. The same potential functions were applied by Rongved[15] to formulate the solution for a force in the interior of two joined semi-infinite solids. These solutions can be obtained by using Muki's formulation as will be shown in the following discussion.

The object of this paper is to apply Muki's formulation to the problem of a layered elastic half space subjected to a concentrated force which may act either vertically or horizontally in the interior of the system. To avoid the use of numerical technique in the evaluation of the integrals involved, accurate approximations of the reciprocals of the common denominators of the integrands by means of integral least square method are suggested. And it will be shown that the proposed solution can be separated into two parts, the first part is the singularity of Mindlin's solution which is singular at the point of application of the force and the second is non-singular and expressed in terms of standard integrals involving Bessel functions. The solution thus obtained is in closed forms. The solution for the plane problems, also in closed forms, are obtained by performing appropriate integrations of the solutions for the corresponding three-dimensional cases.

2. GENERAL EQUATIONS AND SOLUTION

A two-layer elastic half space whose elastic properties, denoted by the shear modulus μ and Poisson's ratio ν, for the upper and lower layers are distinct is depicted in Fig. 1. In this study, four cases for a concentrated force applied either vertically or horizontally in the interior of the upper layer or the lower layer, i.e. the half space, are presented.

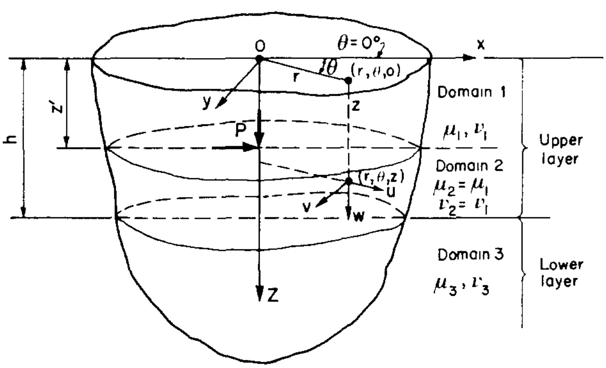

Fig. 1. Force at a point in the interior of a layered elastic half space.

It is convenient to employ cylindrical coordinates (r, θ, z) and to introduce an imaginary horizontal plane passing through the point of application of the concentrated force, which leads to a three-domain problem. The general solution of the Navier displacement equations of equilibrium are obtained by Muki[7] through the utilization of Fourier expansions with respect to the angular coordinate θ and Hankel transforms with respect to the radial coordinate r. Accordingly, the displacements u, v and w may be written as follows:

$$u_i(r, \theta, z) = \sum_{m=0}^{\infty} u_{mi}(r, z)\cos m\theta \tag{1}$$

$$v_i(r, \theta, z) = \sum_{m=0}^{\infty} v_{mi}(r, z)\sin m\theta \tag{2}$$

$$w_i(r, \theta, z) = \sum_{m=0}^{\infty} w_{mi}(r, z)\cos m\theta \tag{3}$$

where the subscript $i = 1$, 2 and 3 denoting the top, middle and bottom domains respectively and, for each harmonic m,

$$\begin{aligned} u_{mi}(r, z) + v_{mi}(r, z) = \int_0^{\infty} \{[\eta^2 A_{mi} + \eta(1 + \eta z)B_{mi} + 2\eta E_{mi}]e^{\eta z} \\ + [-\eta^2 C_{mi} + \eta(1 - \eta z)D_{mi} + 2\eta F_{mi}]e^{-\eta z}\}\eta J_{m+1}(\eta r)\, d\eta \end{aligned} \tag{4}$$

$$\begin{aligned} u_{mi}(r, z) - v_{mi}(r, z) = \int_0^{\infty} \{[-\eta^2 A_{mi} - \eta(1 + \eta z)B_{mi} + 2\eta E_{mi}]e^{\eta z} \\ + [\eta^2 C_{mi} - \eta(1 - \eta z)D_{mi} + 2\eta F_{mi}]e^{-\eta z}\}\eta J_{m-1}(\eta r)\, d\eta \end{aligned} \tag{5}$$

$$\begin{aligned} w_{mi}(r, z) = \int_0^{\infty} \{[-\eta^2 A_{mi} + \eta(2 - 4\nu_i - \eta z)B_{mi}]e^{\eta z} \\ + [-\eta^2 C_{mi} - \eta(2 - 4\nu_i + \eta z)D_{mi}]e^{-\eta z}\}\eta J_m(\eta r)\, d\eta \end{aligned} \tag{6}$$

where J_m is Bessel function of the first kind of order m, the coefficients A_{mi} to F_{mi} are to be determined from the boundary and continuity conditions, and A_{m3}, B_{m3} and E_{m3} are set to be zero in order to take into account the requirement that the functions should be bounded at infinite depth.

Observe that equations (1)–(3) represent only the solution for the case which is symmetric with respect to the coordinate axis $\theta = 0$. For a complete presentation, it is necessary to add the antisymmetric fields which are obtained by interchanging the roles of cos $m\theta$ and sin $m\theta$. The normal stresses σ_r, σ_z and σ_θ, and shearing stresses τ_{zr}, $\tau_{z\theta}$ and $\tau_{r\theta}$ corresponding to the displacements can be obtained from Hooke's law.

3. VERTICAL CONCENTRATED FORCE

Case 1—vertical concentrated force acting in the interior of the upper layer

For the case of a vertical concentrated force P acting downward at a depth z' below the free surface as shown in Fig. 1, the problem is axisymmetric with respect to z-axis. The displacement v, shearing stresses $\tau_{z\theta}$ and $\tau_{r\theta}$, and all the derivatives with respect to θ vanish. In this case equations (1)–(3) are simply given by harmonic terms for $m = 0$ only, and $E_{mi} = F_{mi} = 0$ which is observed from equations (4) and (5). For simplicity, the subscript m will be omitted. The traction due to the concentrated force P can be represented in the form of a Dirac delta function,

$$q(r) = \frac{P\delta(r)}{2\pi r} \tag{7}$$

which may be represented in the integral form

$$q(r) = \int_0^\infty \bar{q}(\eta)\eta J_0(\eta r)\,d\eta \tag{8}$$

where

$$\bar{q}(\eta) = \int_0^\infty q(r) r J_0(\eta r)\,dr = \frac{P}{2\pi}. \tag{9}$$

Boundary and continuity conditions. The free boundary conditions at the plane $z = 0$ are

$$\sigma_{z1}(r, 0) = 0, \qquad \tau_{zr1}(r, 0) = 0. \tag{10}$$

The continuity conditions at the plane $z = z'$ are

$$u_1(r, z') = u_2(r, z'), \qquad w_1(r, z') = w_2(r, z') \tag{11}$$

$$\sigma_{z1}(r, z') - \sigma_{z2}(r, z') = q(r) \tag{12}$$

$$\tau_{zr1}(r, z') = \tau_{zr2}(r, z'). \tag{13}$$

Lastly, the interface continuity conditions at the plane $z = h$ are

$$u_2(r, h) = u_3(r, h), \qquad w_2(r, h) = w_3(r, h) \tag{14}$$

$$\sigma_{z2}(r, h) = \sigma_{z3}(r, h), \qquad \tau_{zr2}(r, h) = \tau_{zr3}(r, h). \tag{15}$$

Substituting the expressions of displacements and stresses into equations (10)–(15), in view of equations (8) and (9), leads to a set of ten simultaneous equations for determining the ten coefficients A_1–D_1, A_2–D_2 and C_3 and D_3. Introducing the dimensionless terms

$$\alpha = \eta h, \qquad \zeta = z/h, \qquad \rho = r/h, \qquad \beta = z'/h, \tag{16}$$

the displacements are in turn obtained in the following dimensionless forms:

$$\frac{\mu_1 h}{P} u_i(\rho, \zeta) = \int_0^\infty \left[\bar{u}_i^*(\alpha, \zeta) + \frac{\bar{u}_i(\alpha, \zeta)}{D(\alpha)}\right] J_1(\rho\alpha)\,d\alpha \tag{17}$$

$$\frac{\mu_1 h}{P} w_i(\rho, \zeta) = \int_0^\infty \left[\bar{w}_i^*(\alpha, \zeta) + \frac{\bar{w}_i(\alpha, \zeta)}{D(\alpha)}\right] J_0(\rho\alpha)\,d\alpha \tag{18}$$

where

$$D(\alpha) = 1 - (a + b + 4b\alpha^2)e^{-2\alpha} + abe^{-4\alpha} \tag{19}$$

$$a = \frac{(3 - 4\nu_3) - \mu_0(3 - 4\nu_1)}{3 - 4\nu_3 + \mu_0} \tag{20}$$

$$b = \frac{(1 - \mu_0)}{1 + \mu_0(3 - 4\nu_1)} \tag{21}$$

$$\mu_0 = \mu_3/\mu_1. \tag{22}$$

For $i = 1$ and 2,

$$\bar{u}_i^*(\alpha, \zeta) = \frac{-1}{16\pi\gamma_0}(\beta - \zeta)\alpha e^{-|\beta-\zeta|\alpha} \tag{23}$$

$$\bar{w}_i^*(\alpha, \zeta) = \frac{1}{16\pi\gamma_0}[\gamma_1 + |\beta - \zeta|\alpha]e^{-|\beta-\zeta|\alpha} \tag{24}$$

$$\begin{aligned}\bar{u}_i(\alpha, \zeta) = \frac{1}{16\pi\gamma_0}&[[-4\gamma_0\gamma_2 - \gamma_1(\beta - \zeta)\alpha + 2\beta\zeta\alpha^2]e^{-(\beta+\zeta)\alpha} + [-\tfrac{1}{2}(a - b\gamma_1^2)\\ &+ b\gamma_1(\beta - \zeta)\alpha - 2b(1 - \beta)(1 - \zeta)\alpha^2]e^{-(2+\beta+\zeta)\alpha} + ab(\beta - \zeta)\alpha e^{-(4-\beta+\zeta)\alpha}\\ &+ [-\tfrac{1}{2}\gamma_1(a - b) + (8b\gamma_0\gamma_2 - b\beta + a\zeta)\alpha - 2b\gamma_1(1 - \beta - \zeta)\alpha^2\\ &+ 4b(1 - \beta)\zeta\alpha^3]e^{-(2-\beta+\zeta)\alpha} + ab(\beta - \zeta)\alpha e^{-(4+\beta-\zeta)\alpha} + [-\tfrac{1}{2}\gamma_1(a - b)\\ &+ (8b\gamma_0\gamma_2 - a\beta + b\zeta)\alpha - 2b\gamma_1(1 - \beta - \zeta)\alpha^2 - 4b\beta(1 - \zeta)\alpha^3]e^{-(2+\beta-\zeta)\alpha}\\ &+ [-4ab\gamma_0\gamma_2 + ab\gamma_1(\beta - \zeta)\alpha + 2ab\beta\zeta\alpha]e^{2-(4-\beta-\zeta)\alpha} - [\tfrac{1}{2}(a - b\gamma_1^2)\\ &+ b\gamma_1(\beta - \zeta)\alpha + 2b(1 - \beta)(1 - \zeta)\alpha^2]e^{-(2-\beta-\zeta)\alpha}]\end{aligned} \tag{25}$$

$$\begin{aligned}w_i(\alpha, \zeta) = \frac{1}{16\pi\gamma_0}&[[(4\gamma_0\gamma_2 + 1) + \gamma_1(\beta + \zeta)\alpha + 2\beta\zeta\alpha^2]e^{-(\beta+\zeta)\alpha} + [-\tfrac{1}{2}(a + b\gamma_1^2)\\ &+ b\gamma_1(2 - \beta - \zeta)\alpha - 2b(1 - \beta)(1 - \zeta)\alpha^2]e^{-(2+\beta+\zeta)\alpha} - ab[\gamma_1\\ &- (\beta - \zeta)\alpha]e^{-(4-\beta+\zeta)\alpha} + \{\tfrac{1}{2}\gamma_1(a + b) + [2b(4\gamma_0\gamma_2 + 1) - b\beta + a\zeta]\alpha\\ &+ 2b\gamma_1(1 - \beta + \zeta)\alpha^2 + 4b(1 - \beta)\zeta\alpha^3\}e^{-(2-\beta+\zeta)\alpha} - ab[\gamma_1\\ &+ (\beta - \zeta)\alpha]e^{-(4+\beta-\zeta)\alpha} + \{\tfrac{1}{2}\gamma_1(a + b) + [2b(4\gamma_0\gamma_2 + 1) + a\beta - b\zeta]\alpha\\ &+ 2b\gamma_1(1 + \beta - \zeta)\alpha^2 + 4b\beta(1 - \zeta)\alpha^3\}e^{-(2+\beta-\zeta)\alpha} + ab[-(4\gamma_0\gamma_2 + 1)\\ &+ \gamma_1(\beta + \zeta)\alpha - 2\beta\zeta\alpha^2]e^{-(4-\beta-\zeta)\alpha} + [\tfrac{1}{2}(a + b\gamma_1^2) + b\gamma_1(2 - \beta - \zeta)\alpha\\ &+ 2b(1 - \beta)(1 - \zeta)\alpha^2]e^{-(2-\beta-\zeta)\alpha}]\end{aligned} \tag{26}$$

and, for $i = 3$,

$$\bar{u}_i^*(\alpha, \zeta) = \bar{w}_i^*(\alpha, \zeta) = 0 \tag{27}$$

$$\begin{aligned}\bar{u}_i(\alpha, \zeta) = \frac{1}{8\pi}&[\{(c - d\gamma_1\gamma_3) + 2[\gamma_1(c - d + d\zeta) - d\beta\gamma_3]\alpha + 4\beta(c - d + d\zeta)\alpha^2\}e^{-(\zeta+\beta)\alpha}\\ &+ \{(-ac + bd\gamma_1\gamma_3) + 2bd[\beta\gamma_3 - 4(v_1 - v_3) - \gamma_1\zeta]\alpha\\ &- 4bd(1 - \beta)(1 - \zeta)\alpha^2\}e^{-(2+\zeta+\beta)\alpha} + \{c\gamma_1 - d\gamma_3 + 2[c(1 - \beta)\\ &- d(1 - \zeta)]\alpha\}e^{-(\zeta-\beta)\alpha} + \{(-ac\gamma_1 + bd\gamma_3) + 2[ac\beta - bd(\gamma_1\gamma_3 - 1 + \zeta)]\alpha\\ &+ 4bd[\beta\gamma_3 - \gamma_1(1 - \zeta)]\alpha^2 + 8bd\beta(1 - \zeta)\alpha^3\}e^{-(2+\zeta-\beta)\alpha}]\end{aligned} \tag{28}$$

$$\begin{aligned}\bar{w}_i(\alpha, \zeta) = \frac{1}{8\pi}&[\{(c + d\gamma_1\gamma_3) + 2[\gamma_1(c - d + d\zeta) + d\beta\gamma_3]\alpha + 4\beta(c - d + d\zeta)\alpha^2\}e^{-(\zeta-\beta)\alpha}\\ &+ \{-(ac + bd\gamma_1\gamma_3) + 2bd[2(3 - 2v_1 - 2v_3) - \beta\gamma_3 - \gamma_1\zeta]\alpha\\ &- 4bd(1 - \beta)(1 - \zeta)\alpha^2\}e^{-(2+\zeta+\beta)\alpha} + \{(c\gamma_1 + d\gamma_3) + 2[c(1 - \beta)\\ &- d(1 - \zeta)]\alpha\}e^{-(\zeta-\beta)\alpha} + \{-(ac\gamma_1 + bd\gamma_3) + 2[ac\beta + bd(\gamma_1\gamma_3 + 1 - \zeta)]\alpha\\ &- 4bd[\beta\gamma_3 + \gamma_1(1 - \zeta)]\alpha^2 + 8bd\beta(1 - \zeta)\alpha^3\}e^{-(2+\zeta-\beta)\alpha}].\end{aligned} \tag{29}$$

In equations (25)–(29),

$$\gamma_0 = 1 - v_1, \qquad \gamma_1 = 3 - 4v_1, \qquad \gamma_2 = 1 - 2v_1, \qquad \gamma_3 = 3 - 4v_3 \tag{30}$$

$$c = \frac{1}{1 + \gamma_1 \mu_0} \tag{31}$$

$$d = \frac{1}{\gamma_3 + \mu_0}. \tag{32}$$

Note that the solution functions take the same form for $i = 1$ and 2. Thus the proposed solution is given in the form of a two-domain problem. The integrals given by $\bar{u}_i^*(\alpha, \zeta)$ and $\bar{w}_i^*(\alpha, \zeta)$ in equations (23) and (24) correspond to the singularity of Mindlin's solution[13], while the integrals given by $\bar{u}_i(\alpha, \zeta)$ and $\bar{w}_i(\alpha, \zeta)$ are non-singular. It is of interest to mention that Mindlin's solution can be derived by the application of the proposed solution technique to the case of a homogeneous half space.

Evaluation of integrals. To avoid the laborious numerical integration[8, 9, 12] in equations (17) and (18), the reciprocal of $D(\alpha)$ involved in these equations will be approximated in such a way that the integrals assume standard forms.

Since $D(\alpha)$ is positive for all values of α from zero to infinite and asymptotic to unity as α approaches infinite, it is reasonable to approximate $1/D(\alpha)$ by a series of exponential functions of the form

$$1/D(\alpha) \simeq 1 + R(\alpha) \tag{33}$$

where

$$R(\alpha) = \sum_{j=1}^{s} k_j \mathrm{e}^{-p_j \alpha} \tag{34}$$

$$k_1 = \frac{a + b - ab}{1 - a - b + ab} - \sum_{j=2}^{s} k_j \tag{35}$$

and s is a positive integer, k_j are constants and p_j are real numbers greater than zero. Observe that the approximate function proposed in equation (33) is smooth and continuous, asymptotically approaches $1/D(\alpha)$ at $\alpha \to \infty$ and in view of equations (19), (34) and (35), assumes the exact value of $1/D(\alpha)$ at $\alpha = 0$. Furthermore, it becomes unity when $\mu_0 = 1$ and $\nu_1 = \nu_3$ and leads exactly to Mindlin's solution.

A logical scheme to determine p_1, k_j and p_j for $j = 2$ to s, is by a process of trial and error with the help of the integral least square method. The normalized error in this case is defined by

$$\varepsilon(\alpha) = 1 - D(\alpha)[1 + R(\alpha)]. \tag{36}$$

The integer s may take a value equal to 4 as a starting point of this approximation as adopted by Burmister[3] for the special case of a rigid lower layer. The values of p_j are first assumed and the constants k_j are then determined in such a way that the square of the error is minimum, i.e.

$$\frac{\partial}{\partial k_j} \int_0^{\infty} [\varepsilon(\alpha)]^2 \, \mathrm{d}\alpha = 0, \qquad j = 2 \text{ to } s. \tag{37}$$

Since the approximation of $1/D(\alpha)$ as proposed yields the exact value of the latter at $\alpha = 0$, the integration of the normal traction on a horizontal plane is equal to zero when the plane is above the point of application of the force P, and equal to P when the plane lies below. If the approximation thus obtained is not accurate enough for practical purposes, another set s and p_j should be assumed for the second trial and so on.

Solution in closed form. In view of the approximation of $1/D(\alpha)$ given by equation (33), the displacements for each layer given by equations (17) and (18) may be expressed in the general form,

$$I = I^* + I^c \tag{38}$$

where

$$I^* = \int_0^\infty \bar{I}^*(\alpha, \zeta) J_m(\rho\alpha)\, \mathrm{d}\alpha \tag{39}$$

$$I^c = \int_0^\infty \bar{I}(\alpha, \zeta)[1 + R(\alpha)] J_m(\rho\alpha)\, \mathrm{d}\alpha. \tag{40}$$

The functions $\bar{I}^*(\alpha, \zeta)$ and $\bar{I}(\alpha, \zeta)$, in view of equations (23)–(29), can be symbolically written in the forms

$$\bar{I}^*(\alpha, \zeta) = \sum_n c_n \alpha^n \mathrm{e}^{-|\beta-\zeta|\alpha} \tag{41}$$

$$\bar{I}(\alpha, \zeta) = \sum_l \sum_n c_{ln} \alpha^n \mathrm{e}^{-f_l \alpha} \tag{42}$$

where the indices l and n are either zero or positive integers, the coefficients c_n and c_{ln} are point functions and f_l is a positive point function. In view of equations (33), (41) and (42), equations (39) and (40) take the forms

$$I^* = \sum_n c_n G[m, n, |\beta - \zeta|] \tag{43}$$

$$I^c = \sum_{j=0}^{s} k_j \left\{ \sum_l \sum_n c_{ln} G[m, n, (f_l + p_j)] \right\} \tag{44}$$

where

$$k_0 = 1, \qquad p_0 = 0 \tag{45}$$

$$G[m, n, p] = \int_0^\infty \alpha^n \mathrm{e}^{-p\alpha} J_m(\rho\alpha)\, \mathrm{d}\alpha \quad \text{for} \quad p > 0. \tag{46}$$

Equation (46) involves standard integrals which are evaluated in closed forms and listed in Appendix A. The integral I^*, which is singular at the point of application of the concentrated force, is the singularity of Mindlin's solution while the integral I^c is non-singular. Thus the proposed solution is in closed forms. In the limiting case of a homogeneous half space, $\mu_0 = 1$, $v_1 = v_3$ and $k_j = 0$ for $j = 1$ to s, and the integral I^c simplifies into the non-singular part of Mindlin's solution.

Case 2—vertical concentrated force acting in the interior of the lower layer

For the case of a vertical concentrated force P applied in the interior of the lower layer at $z = z'$, the half space is divided into three domains: domain 1 for $0 \leqslant z < h$, domain 2 for $h \leqslant z \leqslant z'$ and domain 3 for $z' \leqslant z < \infty$. Thus, $\beta > 1$ for this case. Prescribing the ten boundary and continuity conditions in a similar manner as in Case 1 leads to the solution which assumes the same form given in equations (17) and (18). In this case, for $i = 1$,

$$\bar{u}_i^*(\alpha, \zeta) = \bar{w}_i^*(\alpha, \zeta) = 0 \tag{47}$$

$$\bar{u}_i(\alpha, \zeta) = \frac{1}{8\pi}\big[\{\gamma_1 c - \gamma_3 d - 2[c(1-\zeta) + d(\beta - 1)]\alpha\}e^{-(\beta-\zeta)\alpha} + \{-\gamma_1 ac + \gamma_3 bd + 2[bd(\gamma_1\gamma_3 - 1 + \beta) - ac\zeta]\alpha + 4bd[\gamma_3\zeta + \gamma_1(\beta - 1)]\alpha^2 + 8bd(\beta - 1)\zeta\alpha^3\}e^{-(\beta+2-\zeta)\alpha} + \{c - \gamma_1\gamma_3 d + [-2\gamma_1(c + d\beta - d) + 2d\gamma_3\zeta]\alpha + 4(c + d\beta - d)\zeta\alpha^2\}e^{-(\beta+\zeta)\alpha} + \{-ac + bd\gamma_1\gamma_3 + 2bd[\gamma_3(1-\zeta) + \gamma_1(\beta - 1)]\alpha + 4bd(\beta - 1)(1-\zeta)\alpha^2\}e^{-(\beta+2+\zeta)\alpha}\big] \tag{48}$$

$$\bar{w}_i(\alpha, \zeta) = \frac{1}{8\pi}\big[\{\gamma_1 c + \gamma_3 d + 2[c(1-\zeta) + d(\beta - 1)]\alpha\}e^{-(\beta-\zeta)\alpha} + \{-(\gamma_1 ac + bd\gamma_3) + 2[bd(\gamma_1\gamma_3 - \beta + 1) + ac\zeta]\alpha + 4bd[\gamma_1(\beta - 1) - \gamma_3\zeta]\alpha^2 - 8bd(\beta - 1)\zeta\alpha^3\}e^{-(\beta+2-\zeta)\alpha} + \{c + \gamma_1\gamma_3 d + 2[\gamma_1(c + d\beta - d) + d\gamma_3\zeta]\alpha + 4(c + d\beta - d)\zeta\alpha^2\}e^{-(\beta+\zeta)\alpha} + \{-(ac + bd\gamma_1\gamma_3) + 3bd[\gamma_3(1-\zeta) - \gamma_1(\beta - 1)]\alpha + 4bd(\beta - 1)(1-\zeta)\alpha^2\}e^{-(\beta+2+\zeta)\alpha}\big] \tag{49}$$

and, for $i = 2$ and 3,

$$\bar{u}_i^*(\alpha, \zeta) = \frac{1}{16\pi\mu_0(1-\nu_3)}\left[-(\beta-\zeta)\alpha e^{-|\beta-\zeta|\alpha} + \left\{\frac{\gamma_3}{2}[(1-\mu_0)d\gamma_3 - 1] + (1-\mu_0)d\gamma_3(\beta-\zeta)\alpha + 2(1-\mu_0)d(\beta-1)(1-\zeta)\alpha^2\right\}e^{-(\beta-2+\zeta)\alpha}\right] \tag{50}$$

$$\bar{w}_i^*(\alpha, \zeta) = \frac{1}{16\pi\mu_0(1-\nu_3)}\left[[\gamma_3 + |\beta-\zeta|\alpha]e^{-|\beta-\zeta|\alpha} + \left\{-\frac{\gamma_3}{2}[1 + (1-\mu_0)d\gamma_3] + (1-\mu_0)d\gamma_3(2-\beta-\zeta)\alpha + 2(1-\mu_0)d(\beta-1)(1-\zeta)\alpha^2\right\}e^{-(\beta-2+\zeta)\alpha}\right] \tag{51}$$

$$\bar{u}_i(\alpha, \zeta) = \frac{1}{8\pi}\big[\gamma_1 c e^{-(\beta-2+\zeta)\alpha} + \{c - \gamma_1 ac - 4\gamma_0\gamma_3^2 d^2 - 8\gamma_0\gamma_3 d^2(\beta-\zeta)\alpha + 4[c + 4cd\gamma_0(\beta-1) - 4\gamma_0 d(c + d\beta - d)(1-\zeta)]\alpha^2\{e^{-(\beta+\zeta)\alpha} + [-ac + 4\gamma_0\gamma_3^2 bd^2 + 8\gamma_0 bd^2\gamma_3(\beta-\zeta)d + 16\gamma_0 bd^2(\beta-1)(1-\zeta)\alpha^2]e^{-(\beta+2+\zeta)\alpha}\big] \tag{52}$$

$$\bar{w}_i(\alpha, \zeta) = \frac{1}{8\pi}\big[\gamma_1 c e^{-(\beta-2+\zeta)\alpha} + \{c - \gamma_1 ac + 4\gamma_0\gamma_3^2 d^2 + 8\gamma_0\gamma_3 d[2c + d(\beta + \zeta - 2)]\alpha + 4[c + 4cd\gamma_0(\beta-1) - 4\gamma_0 d(c + d\beta - d)(1-\zeta)]\alpha^2\}e^{-(\beta+\zeta)\alpha} + [-(ac + 4\gamma_0\gamma_3^2 bd^2) + 8\gamma_0\gamma bd^2(2-\beta-\zeta)\alpha + 16\gamma_0 bd^2(\beta-1)(1-\zeta)\alpha^2]e^{-(\beta+2+\zeta)\alpha}\big]. \tag{53}$$

It should be mentioned that $1/D(\alpha)$ is singular at $\alpha = 0$ when $\mu_0 = 0$, which is immaterial for Case 1 but a practical limiting condition for Case 2. For convenience in the treatment of the latter, the solution for Case 2 is formulated in such a way that $\bar{u}_i^*(\alpha, \zeta)$ and $\bar{w}_i^*(\alpha, \zeta)$ contains singular as well as non-singular terms. For $\mu_0 = 0$, the case of a rigid upper layer, the functions $\bar{u}_i(\alpha, \zeta)$ and $\bar{w}_i(\alpha, \zeta)$ of the lower layer vanish and the solution, given by $\bar{u}_i^*(\alpha, \zeta)$ and $\bar{w}_i^*(\alpha, \zeta)$ only, is exact. Obviously this special case is identical to the half space fixed on the surface, a limiting case of Rongved's solution.

4. HORIZONTAL CONCENTRATED FORCE

Case 3—horizontal concentrated force acting in the interior of the upper layer

For the case of a horizontal concentrated force P applied at a depth z' below the free surface as shown in Fig. 1, the problem is symmetric with respect to the coordinate axis $\theta = 0$, oriented in the direction of the force, and the traction due to the concentrated force P is represented in the form of equation (8).

Boundary and continuity conditions. The free boundary conditions at the plane $z = 0$ are

$$\sigma_{z1}(r, \theta, 0) = 0 \tag{54}$$

$$\tau_{z\theta 1}(r, \theta, 0) = 0, \qquad \tau_{zr1}(r, \theta, 0) = 0. \tag{55}$$

The continuity conditions at the plane $z = z'$ are

$$u_1(r, \theta, z') = u_2(r, \theta, z'), \qquad v_1(r, \theta, z') = v_2(r, \theta, z') \tag{56}$$

$$w_1(r, \theta, z') = w_2(r, \theta, z') \tag{57}$$

$$\sigma_{z1}(r, \theta, z') = \sigma_{z2}(r, \theta, z') \tag{58}$$

$$\tau_{z\theta 1}(r, \theta, z') - \tau_{z\theta 2}(r, \theta, z') = -q(r)\sin\theta \tag{59}$$

$$\tau_{zr1}(r, \theta, z') - \tau_{zr2}(r, \theta, z') = q(r)\cos\theta. \tag{60}$$

Lastly, the interface continuity conditions at the plane $z = h$ are

$$u_2(r, \theta, h) = u_3(r, \theta, h), \qquad v_2(r, \theta, h) = v_3(r, \theta, h) \tag{61}$$

$$w_2(r, \theta, h) = w_3(r, \theta, h) \tag{62}$$

$$\sigma_{z2}(r, \theta, h) = \sigma_{z3}(r, \theta, h) \tag{63}$$

$$\tau_{z\theta 2}(r, \theta, h) = \tau_{z\theta 3}(r, \theta, h), \qquad \tau_{zr2}(r, \theta, h) = \tau_{zr3}(r, \theta, h). \tag{64}$$

In view of equations (59) and (60), the displacements and stresses are simply given by the harmonic terms for $m = 1$ only. For simplicity, the subscript m will be omitted. The set of fifteen coefficients A_1–F_1, A_2–F_2, and C_3, D_3 and F_3 are to be determined from equations (54)–(64). To facilitate this task, u_i and v_i are rewritten in combined forms as equations (4) and (5) and the shearing stresses $\tau_{z\theta i}$ and τ_{zri} are treated in the same manner[8]. In this case the dimensionless forms of the displacements are given by

$$\frac{\mu_1 h}{P\cos\theta} u_i(\rho, \theta, \zeta) = \tfrac{1}{2}[U_i(\rho, \zeta) + V_i(\rho, \zeta)] \tag{65}$$

$$\frac{\mu_1 h}{P\sin\theta} v_i(\rho, \theta, \zeta) = \tfrac{1}{2}[U_i(\rho, \zeta) - V_i(\rho, \zeta)] \tag{66}$$

$$\frac{\mu_1 h}{P\cos\theta} w_i(\rho, \theta, \zeta) = \int_0^\infty \left[\bar{w}_i^*(\rho, \zeta) + \frac{\bar{w}_i(\rho, \zeta)}{D(\alpha)}\right] J_1(\rho\alpha)\, \mathrm{d}\alpha \tag{67}$$

where

$$U_i(\rho, \zeta) = \int_0^\infty [\bar{U}_i^*(\alpha, \zeta) + \bar{V}_i^*(\alpha, \zeta)] + \left[\frac{\bar{U}_i(\alpha, \zeta)}{D(\alpha)} + \frac{\bar{V}_i(\alpha, \zeta)}{H(\alpha)}\right] J_2(\rho\alpha)\, \mathrm{d}\alpha \tag{68}$$

$$V_i(\rho,\zeta)=\int_0^\infty\left\{[-\overline{U}_i^*(\alpha,\zeta)+\overline{V}_i^*(\alpha,\zeta)]+\left[\frac{-\overline{U}_i(\alpha,\zeta)}{D(\alpha)}+\frac{\overline{V}_i(\alpha,\zeta)}{H(\alpha)}\right]\right\}J_0(\rho\alpha)\,\mathrm{d}\alpha \tag{69}$$

$$H(\alpha)=1-\frac{1-\mu_0}{1+\mu_0}\mathrm{e}^{-2\alpha}. \tag{70}$$

For $i=1$ and 2

$$\overline{U}_i^*(\alpha,\zeta)=\frac{1}{16\pi\gamma_0}[-\gamma_1+(\beta-\zeta)\alpha]\mathrm{e}^{-|\beta-\zeta|\alpha} \tag{71}$$

$$\overline{V}_i^*(\alpha,\zeta)=\frac{1}{4\pi}\mathrm{e}^{-|\beta-\zeta|\alpha} \tag{72}$$

$$\overline{w}_i^*(\alpha,\zeta)=\frac{1}{16\pi\gamma_0}(\beta-\zeta)\alpha\mathrm{e}^{-|\beta-\zeta|\alpha} \tag{73}$$

$$\begin{aligned}\overline{U}_i(\alpha,\zeta)=\frac{1}{16\pi\gamma_0}\Big[&[-(4\gamma_0\gamma_2+1)+\gamma_1(\beta+\zeta)\alpha-2\beta\zeta\alpha^2]\mathrm{e}^{-(\beta+\zeta)\alpha}+[\tfrac{1}{2}(a+b\gamma_1^2)\\&+b\gamma_1(2-\beta-\zeta)\alpha+2b(1-\beta)(1-\zeta)\alpha^2[\mathrm{e}^{-(2+\beta+\zeta)\alpha}+ab[\gamma_1+(\beta-\zeta)\alpha]\mathrm{e}^{-(4-\beta+\zeta)\alpha}\\&+\Big\{-\frac{\gamma_1}{2}(a+b)+[a\zeta+b(8\gamma_0\gamma_1+2-\beta)]\alpha-2b\gamma_1(1-\beta+\zeta)\alpha^2\\&+4b(1-\beta)\zeta\alpha^3\Big\}\mathrm{e}^{-(2-\beta+\zeta)\alpha}+ab[\gamma_1-(\beta-\zeta)\alpha]\mathrm{e}^{-(4+\beta-\zeta)\alpha}+\Big\{-\frac{\gamma_1}{2}(a+b)\\&+[a\beta+2b(4\gamma_0\gamma_2+1)-b\zeta]\alpha-2b\gamma_1(1+\beta-\zeta)\alpha^2+4b\beta(1-\zeta)\alpha^3\Big\}\mathrm{e}^{-(2+\beta-\zeta)\alpha}\\&+ab[(4\gamma_0\gamma_2+1)+\gamma_1(\beta+\zeta)\alpha+2\beta\zeta\alpha^2]\mathrm{e}^{-(4-\beta-\zeta)\alpha}+[-\tfrac{1}{2}(a+b\gamma_1^2)\\&+b\gamma_1(2-\beta-\zeta)\alpha-2b(1-\beta)(1-\zeta)\alpha^2]\mathrm{e}^{-(2-\beta-\zeta)\alpha}\Big]\end{aligned} \tag{74}$$

$$\overline{V}_i(\alpha,\zeta)=\frac{1}{4\pi}\left[\mathrm{e}^{-(\beta+\zeta)\alpha}+\frac{1-\mu_0}{1+\mu_0}\{\mathrm{e}^{-(2-\beta+\zeta)\alpha}+\mathrm{e}^{-(2+\beta-\zeta)\alpha}+\mathrm{e}^{-(2-\beta-\zeta)\alpha}\}\right] \tag{75}$$

$$\begin{aligned}\overline{w}_i(\alpha,\zeta)=\frac{1}{16\pi\gamma_0}\Big[&[4\gamma_0\gamma_2-\gamma_1(\beta-\zeta)\alpha-2\beta\zeta\alpha^2]\mathrm{e}^{-(\beta+\zeta)\alpha}+[\tfrac{1}{2}(a-b\gamma_1^2)\\&+b\gamma_1(\beta-\zeta)\alpha+2b(1-\beta)(1-\zeta)\alpha^2]\mathrm{e}^{-(2+\beta+\zeta)\alpha}+ab(\beta-\zeta)\alpha\mathrm{e}^{-(4-\beta+\zeta)\alpha}\\&+\Big\{\frac{\gamma_2}{2}(a-b)+[-b(\beta+8\gamma_0\gamma_2)+a\zeta]\alpha+2b\gamma_1(1-\beta-\zeta)\alpha^2+4b(1\\&-\beta)\zeta\alpha^3\Big\}\mathrm{e}^{-(2-\beta+\zeta)\alpha}+ab(\beta-\zeta)\alpha\mathrm{e}^{-(4+\beta-\zeta)\alpha}+\Big[\frac{\gamma_1}{2}(a-b)+(b\zeta-a\beta\\&+8\gamma_0\gamma_2)\alpha+2b\gamma_1(1-\beta-\zeta)\alpha^2-4b\beta(1-\zeta)\alpha^3\Big]\mathrm{e}^{-(2+\beta-\zeta)\alpha}+ab[4\gamma_0\gamma_2\\&+\gamma_1(\beta-\zeta)\alpha-2\beta\zeta\alpha^2[\mathrm{e}^{-(4-\beta-\zeta)\alpha}+[\tfrac{1}{2}(a-b\gamma_1^2)-b\gamma_1(\beta-\zeta)\alpha\\&+2b(1-\beta)(1-\zeta)\alpha^2]\mathrm{e}^{-(2-\beta-\zeta)\alpha}\Big]\end{aligned} \tag{76}$$

and, for $i = 3$,

$$\bar{U}^*(\alpha, \zeta) = \bar{V}^*(\alpha, \zeta) = \bar{w}_i^*(\alpha, \zeta) = 0. \tag{77}$$

$$\begin{aligned}\bar{U}_i(\alpha, \zeta) = \frac{1}{8\pi} [\{-(c + d\gamma_1\gamma_3) + 2[\gamma_1(c - d + d\zeta) + d\beta\gamma_3]\alpha - 4\beta(c - d + d\zeta)\alpha^2\}e^{-(\zeta+\beta)\alpha} \\ + \{(ac - bd\gamma_1\gamma_3) + 2bd[\gamma_1(1 - \zeta) + \gamma_3(1 - \beta)]\alpha + 4bd(1 - \beta)(1 \\ - \zeta)\alpha^2\} e^{-(2+\zeta+\beta)\alpha} + \{(-c\gamma_1 + d\gamma_3) + 2[c(1 - \beta) - d(1 - \zeta)]\alpha\}e^{-(\zeta-\beta)\alpha} \\ + \{(ac\gamma_1 - bd\gamma_3) + 2[ac\beta + bd(\gamma_1\gamma_3 + 1 - \zeta)]\alpha + 4bd[\gamma_1(1 - \zeta) \\ + \gamma_3\beta]\alpha^2 + 8bd\beta(1 - \zeta)\alpha^3\}e^{-(2+\zeta-\beta)\alpha}]\end{aligned} \tag{78}$$

$$\bar{V}_i(\alpha, \zeta) = \frac{\mu_0}{2\pi(1 + \mu_0)} [e^{-(\zeta+\beta)\alpha} + e^{-(\zeta-\beta)\alpha}] \tag{79}$$

$$\begin{aligned}\bar{w}_i(\alpha, \zeta) = \frac{1}{8\pi} [\{-c + d\gamma_1\gamma_3 + 2[\gamma_1(c - d + d\zeta) - d\beta\gamma_3]\alpha - 4\beta(c - d + d\zeta)\alpha^2\}e^{-(\zeta+\beta)\alpha} \\ + \{(ac - bd\gamma_1\gamma_3) + 2bd[\gamma_1(1 - \zeta) - \gamma_3(1 - \beta)]\alpha + 4bd(1 - \beta)(1 - \zeta)\alpha^2\}e^{-(2+\zeta+\beta)\alpha} \\ + \{(-c\gamma_1 + d\gamma_3) + 2[c(1 - \beta) - d(1 - \zeta)]\alpha\}e^{-(\zeta-\beta)\alpha} + \{(ac\gamma_1 - bd\gamma_3) + 2[ac\beta \\ - bd(\gamma_1\gamma_3 - 1 + \zeta)]\alpha + 4bd[\gamma_1(1 - \zeta) - \gamma_3\beta]\alpha^2 + 8bd\beta(1 - \zeta)\alpha^3\}e^{-(2+\zeta-\beta)\alpha}].\end{aligned} \tag{80}$$

The denominator $D(\alpha)$ takes the same form as equation (19). Again the proposed solution is given in the form of a two-domain problem. The integrals given by $\bar{U}^*(\alpha, \zeta)$, $\bar{V}^*(\alpha, \zeta)$ and $\bar{w}^*(\alpha, \zeta)$ correspond to the singularity of Mindlin's solution for a horizontal concentrated force. The total solution of the latter can be obtained as a limiting case.

Evaluation of integrals. Following the same arguments used in the approximation of $1/D(\alpha)$, $1/H(\alpha)$ will be approximated by

$$\frac{1}{H(\alpha)} \simeq 1 + R'(\alpha) \tag{81}$$

where

$$R'(\alpha) = \sum_{j=1}^{s'} k'_j e^{-p_j'\alpha} \tag{82}$$

$$k'_1 = \frac{1 - \mu_0}{2\mu_0} - \sum_{j=2}^{s'} k'_j. \tag{83}$$

In view of the approximations of $1/D(\alpha)$ and $1/H(\alpha)$, the integrals in equations (65) and (66) may be expressed in the same general form as equation (38) with I^* and I^c taking the forms

$$I^* = \int_0^\infty [\bar{I}_1^*(\alpha, \zeta) + \bar{I}_2^*(\alpha, \zeta)]J_m(\rho\alpha)\, d\alpha \tag{84}$$

$$I^c = \int_0^\infty \{\bar{I}_1(\alpha, \zeta)[1 + R(\alpha)] + \bar{I}_2(\alpha, \zeta)[1 + R'(\alpha)]\}J_m(\rho\alpha)\, d\alpha. \tag{85}$$

As in Case 1, the singular integral I^* and the non-singular integral I^c are given in closed forms of finite series of standard integrals $G[m, n, p]$ as defined in equation (46) and listed in Appendix A.

Case 4—horizontal concentrated force acting in the interior of the lower layer

For the case of a horizontal concentrated force P applied in the interior of the lower layer at $z = z'$, the half space is divided into three domains as in Case 2. Prescribing the fifteen boundary and continuity conditions in a similar manner as in Case 3 leads to the solution which assumes the same form given in equations (65)–(67). In this case, for $i = 1$,

$$\bar{U}_i^*(\alpha, \zeta) = \bar{V}_i^*(\alpha, \zeta) = \bar{w}_i^*(\alpha, \zeta) = 0 \tag{86}$$

$$\begin{aligned}\bar{U}_i(\alpha, \zeta) = \frac{1}{4\pi}[&\{-\tfrac{1}{2}(\gamma_1 c + \gamma_3 d) + [c(1-\zeta) + d(\beta - 1)]\alpha\}e^{-(\beta-\zeta)\alpha} + \{\tfrac{1}{2}(\gamma_1 ac + \gamma_3 bd)\\ &+ [bd(\gamma_1\gamma_3 - \beta + 1) + ac\zeta]\alpha + 2bd(\gamma_3\zeta - \gamma_1)\alpha^2 - 4bd(\beta-1)\zeta\alpha^3\}e^{-(\beta+2-\zeta)\alpha}\\ &+ \{-\tfrac{1}{2}(c + \gamma_1\gamma_3 d) + [\gamma_1(c - d + d\beta) + \gamma_3 d\zeta]\alpha - 2[c + d(\beta-1)]\zeta\alpha^2\}e^{-(\beta+\zeta)\alpha}\\ &+ \{\tfrac{1}{2}(ac + \gamma_1\gamma_3 bd) + bd[\gamma_3(1-\zeta) - \gamma_1(\beta-1)]\alpha - 2bd(\beta-1)(1-\zeta)\alpha^2\}e^{-(\beta+2+\zeta)\alpha}].\end{aligned} \tag{87}$$

$$\bar{V}_i(\alpha, \zeta) = \frac{1}{2\pi(1+\mu_0)}[e^{-(\beta-\zeta)\alpha} + e^{-(\beta+\zeta)\alpha}] \tag{88}$$

$$\begin{aligned}\bar{w}_i(\alpha, \zeta) = \frac{1}{4\pi}[&\{\tfrac{1}{2}(-\gamma_1 c + \gamma_3 d) - [c(1-\zeta) + d(\beta-1)]\alpha\}e^{-(\beta-\zeta)\alpha} + \{\tfrac{1}{2}(\gamma_1 ac - \gamma_3 bd)\\ &+ [bd(\gamma_1\gamma_3 + \beta - 1) - ac\zeta]\alpha - 2bd[\gamma_1(\beta-1) + \gamma_3\zeta]\alpha^2 + 4bd(\beta-1)\zeta\alpha^3\}e^{-(\beta+2-\zeta)\alpha}\\ &+ \{\tfrac{1}{2}(-c + \gamma_1\gamma_3 d) + [-\gamma_1(c - d + d\beta) + \gamma_3 d\zeta]\alpha - 2[c + d(\beta-1)] + \zeta\alpha^2\}e^{-(\beta+\zeta)\alpha}\\ &+ \tfrac{1}{2}(ac - \gamma_1\gamma_3 bd) + bd[\gamma_3(1-\zeta) + \gamma_1(\beta-1)]\alpha - 2bd(\beta-1)(1-\zeta)\alpha^2\}e^{-(\beta+2+\zeta)\alpha}]\end{aligned} \tag{89}$$

and, for $i = 2$ and 3,

$$\begin{aligned}\bar{U}_i^*(\alpha, \zeta) = \frac{1}{16\pi\mu_0(1-\nu_3)}\Bigg[&[-\gamma_3 + (\beta-\zeta)\alpha]e^{-|\beta-\zeta|\alpha} + \Bigg\{\frac{\gamma_3}{2}[1 + \gamma_3(1-\mu_0)d]\\ &+ (1-\mu_0)d\gamma_3(2-\beta-\zeta)\alpha - 2(1-\mu_0)d(\beta-1)(1-\zeta)\alpha^2\Bigg\}e^{-(\beta-2+\zeta)\alpha}\Bigg]\end{aligned} \tag{90}$$

$$\bar{V}_i^*(\alpha, \zeta) = \frac{1}{4\pi}[e^{-|\beta-\zeta|\alpha} - e^{-(\beta-2+\zeta)\alpha}] \tag{91}$$

$$\begin{aligned}\bar{w}_i^*(\alpha, \zeta) = \frac{1}{16\pi\mu_0(1-\nu_3)}\Bigg[&-(\beta-\zeta)\alpha e^{-|\beta-\zeta|\alpha} + \Bigg\{\frac{\gamma_3}{2}[1 - \gamma_3(1-\mu_0)d]\\ &+ (1-\mu_0)d\gamma_3(\beta-\zeta)\alpha - 2(1-\mu_0)d(\beta-1)(1-\zeta)\alpha^2\Bigg\}e^{-(\beta-2+\zeta)\alpha}\Bigg]\end{aligned} \tag{92}$$

$$\bar{U}_i(\alpha, \zeta) = \frac{1}{4\pi}[-\tfrac{1}{2}\gamma_1 c e^{-(\beta-2+\zeta)\alpha} + (-\tfrac{1}{2}(c - \gamma_1 ac + 4\gamma_0\gamma_3^2 d^2) + 4\gamma_0\gamma_3 d(2c - 2 + \beta + \zeta)\alpha$$
$$- 2\{c + 4\gamma_0 d[c(\beta - 1) - (c - d + d\beta)(1 - \zeta)]\}\alpha^2)e^{-(\beta+\zeta)\alpha} + [\tfrac{1}{2}(ac + 4\gamma_0\gamma_3^2 bd^2)$$
$$+ 4\gamma_0\gamma_3 bd^2(2 - \beta - \zeta)\alpha - 8\gamma_0 bd^2(\beta - 1)(1 - \zeta)\alpha^2]e^{-(\beta+2+\zeta)\alpha}] \tag{93}$$

$$\bar{V}_i(\alpha, \zeta) = \frac{\mu_0}{2\pi(1 + \mu_0)}[e^{-(\beta+\zeta)\alpha} + e^{-(\beta-2+\zeta)\alpha}] \tag{94}$$

$$\bar{w}_i(\alpha, \zeta) = \frac{1}{4\pi}[-\tfrac{1}{2}\gamma_1 c e^{-(\beta-2+\zeta)\alpha} + (\tfrac{1}{2}(-c + \gamma_1 ac + 4\gamma_0\gamma_3^2 d^2) - 4\gamma_0\gamma_3 d^2(\beta - \zeta)\alpha$$
$$- 2\{c + 4\gamma_0 d[c(\beta - 1) - (c - d + d\beta)(1 - \zeta)]\}\alpha^2)e^{-(\beta+\zeta)\alpha} + [\tfrac{1}{2}(ac - 4\gamma_0\gamma_3^2 bd^2)$$
$$+ 4\gamma_0 bd^2\gamma_3(\beta - \zeta)\alpha - 8\gamma_0 bd^2(\beta - 1)(1 - \zeta)\alpha^2]e^{-(\beta+2+\zeta)\alpha}]. \tag{95}$$

It should be mentioned that, in this case, $1/D(\alpha)$ and $1/H(\alpha)$ are singular at $\alpha = 0$ when $\mu_0 = 0$, a practical limiting case. The solution is formulated in the same manner as Case 2, and $\bar{U}_i^*(\alpha, \zeta)$, $\bar{V}_i^*(\alpha, \zeta)$ and $\bar{w}_i^*(\alpha, \zeta)$ contain singular as well as non-singular terms. For $\mu_0 = 0$, the functions $\bar{U}_i(\alpha, \zeta)$, $\bar{V}_i(\alpha, \zeta)$ and $\bar{w}_i(\alpha, \zeta)$ of the lower layer vanish, and the solution, given by $\bar{U}_i^*(\alpha, \zeta)$, $\bar{V}_i^*(\alpha, \zeta)$ and $\bar{w}_i^*(\alpha, \zeta)$ only, is exact. The latter is the solution of the half space fixed on the surface, a limiting case of Rongved's solution.

5. PLANE PROBLEMS

For the plane strain problem of a vertical line force uniformly distributed on a line parallel to y-axis, the solution can be obtained from that for a concentrated force P by performing an appropriate integration[16]. The normal stresses σ_{zi} and σ_{xi} and shearing stresses τ_{zxi} are obtained with dimensionless forms,

$$\frac{h^2}{4P}\sigma_{zi}(\xi, \zeta) = \int_0^\infty \left[\bar{\sigma}_{zi}^*(\alpha, \zeta) + \frac{\bar{\sigma}_{zi}(\alpha, \zeta)}{D(\alpha)}\right]\frac{\cos\xi\alpha}{\alpha}\,d\alpha \tag{96}$$

$$\frac{h^2}{4P}\tau_{zxi}(\xi, \zeta) = \int_0^\infty \left[\bar{\tau}_{zri}^*(\alpha, \zeta) + \frac{\bar{\tau}_{zri}(\alpha, \zeta)}{D(\alpha)}\right]\frac{\sin\xi\alpha}{\alpha}\,d\alpha \tag{97}$$

$$\frac{h^2}{4P}\sigma_{xi}(\xi, \zeta) = \int_0^\infty \left[\bar{\sigma}_{ri}^*(\alpha, \zeta) + \frac{\bar{\sigma}_{ri}(\alpha, \zeta)}{D(\alpha)}\right]\frac{\cos\xi\alpha}{\alpha}\,d\alpha \tag{98}$$

where $\bar{\sigma}_{zi}^*(\alpha, \zeta)$, $\bar{\tau}_{zri}^*(\alpha, \zeta)$, $\bar{\sigma}_{ri}^*(\alpha, \zeta)$, $\bar{\sigma}_{zi}(\alpha, \zeta)$, $\bar{\tau}_{zri}(\alpha, \zeta)$ and $\bar{\sigma}_{ri}(\alpha, \zeta)$ are functions obtained from the corresponding solution for a concentrated force P, and

$$\xi = x/h. \tag{99}$$

In view of the approximation of $1/D(\alpha)$, equations (97)–(98) may be expressed in the general form of equation (38). In this case, the integrals I^* and I^c are given by finite series of standard integrals $G_s[n, p]$ and $G_c[n, p]$ in the forms

$$G_s[n, p] = \int_0^\infty \alpha^{(n-1)}e^{-p\alpha}\sin\xi\alpha\,d\alpha \tag{100}$$

$$G_c[n, p] = \int_0^\infty \alpha^{(n-1)}e^{-p\alpha}\cos\xi\alpha\,d\alpha \tag{101}$$

in which $n = 0$ and positive integers, and $p > 0$. The closed form solutions (100) and (101) are listed in Appendix A.

For the case of a horizontal line force, the solution of the plane strain problem is obtained in similar forms as given by equations (96)–(98) with the substitutions of $(-\cos \xi\alpha)$ and $\sin \xi\alpha$ for $\sin \xi\alpha$ and $\cos \xi\alpha$ respectively.

The solution of a plane strain problem with Poisson's ration ν can be taken as the solution for a plane stress problem with Poisson's ratio equal to $\nu/(1-\nu)$. It is of interest to note that equations (96)–(98) yield Melan's solution[17] as a limiting case of a homogeneous half plane under plane stress condition.

The displacements in these plane problems, as in any plane problems involving an infinite elastic domain, are arbitrary.

6. NUMERICAL RESULTS AND DISCUSSION

For any values of ν_1 and ν_3, $1/D(\alpha)$ is greater than unity when $\mu_0 < 1$; and less than unity but greater than zero when $\mu_0 > 1$. This is shown in Fig. 2 for $\nu_1 = \nu_3 = 1/4$ and $\mu_0 = 0{\cdot}02$, $0{\cdot}1$ and $5{\cdot}0$ as examples. Furthermore, for $\mu_0 < 1$, the value of $1/D(0)$ increases as μ_0 decreases. This information is very helpful in assuming the set of p_j in the approximation scheme. For $\nu_1 = \nu_3 = 1/4$, and $\mu_0 = 0{\cdot}02$, $0{\cdot}1$, $0{\cdot}5$, $5{\cdot}0$, $10{\cdot}0$ and ∞, the approximations of $1/D(\alpha)$ and $1/H(\alpha)$ are given in Appendix B. For other values of Poisson's ratio, the approximations are presented in [16].

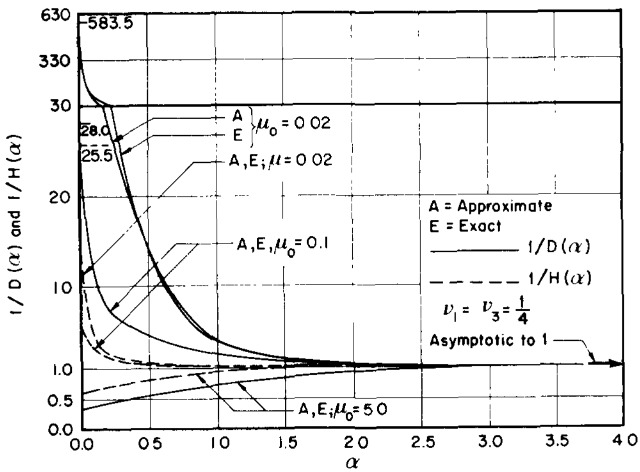

Fig. 2. Comparison of the reciprocals of the common denominators with their approximations.

The comparison of $1/D(\alpha)$ and $1/H(\alpha)$ with their approximations over the whole domain of interest, $0 \leqslant \alpha < \infty$, as shown in Fig. 2, is quite satisfactory for all cases. These approximations may lead to slight violation of the boundary and continuity conditions, but the discrepancies are not significant for practical proposes as shown, for examples, in Figs. 3–8. The results for the displacements yield higher order of accuracy.

For the case of a vertical concentrated force applied at the mid-depth of the upper layer with $\mu_0 = 5{\cdot}0$, Figs. 4 and 5 show that the stresses σ_z and τ_{zr} are continuous at the interface and damp out slowly inside the lower layer. Figure 5 shows a double-peak distribution of τ_{zr} at different radii near the horizontal plane through the point of application of the force, which is due to the high concentration of stresses in the vicinity of the force where the

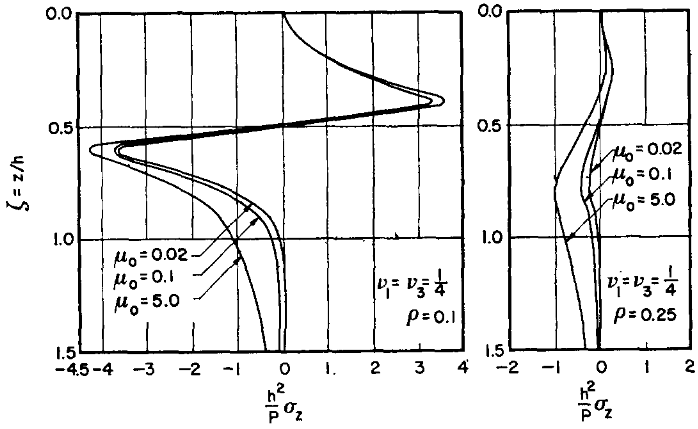

Fig. 3. Variation of σ_z for Case 1, $\rho = 0{\cdot}1$ and $0{\cdot}25$, $\beta = 0{\cdot}5$, $\mu_0 = 0{\cdot}02$, $0{\cdot}1$ and $5{\cdot}0$.

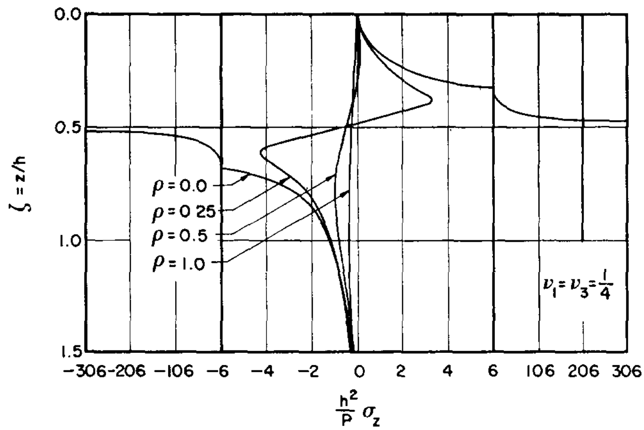

Fig. 4. Distribution of σ_z for Case 1, $\beta = 0{\cdot}5$ and $\mu_0 = 5{\cdot}0$.

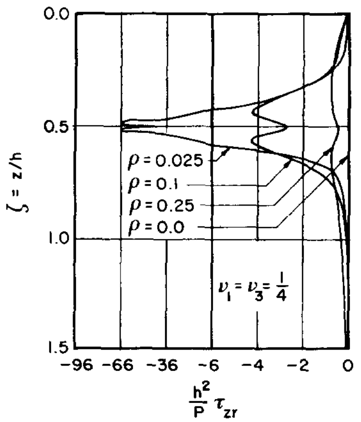

Fig. 5. Distribution of τ_{zr} for Case 1, $\beta = 0{\cdot}5$ and $\mu_0 = 5{\cdot}0$.

singularity of Mindlin's solution is dominant. Another set of numerical results is presented in Figs. 6–8 for the case of a horizontal concentrated force applied at the mid-depth of the upper layer with $\mu_0 = 0{\cdot}02$. The stresses in the lower layer are small due to the low ratio of μ_0.

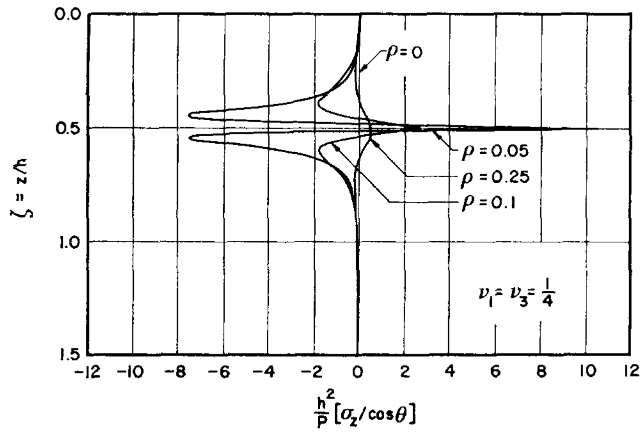

Fig. 6. Distribution of σ_z for Case 3, $\beta = 0{\cdot}5$ and $\mu_0 = 0{\cdot}02$.

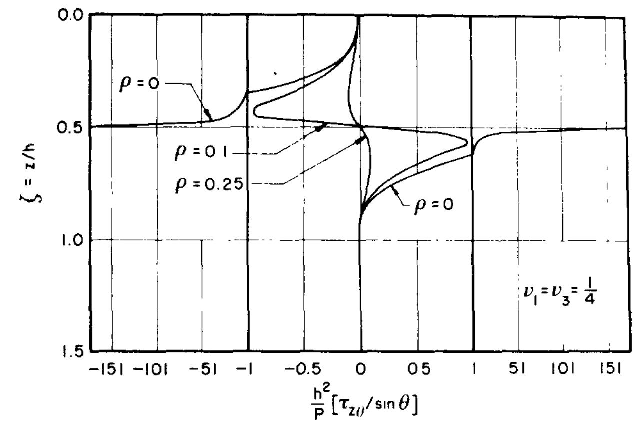

Fig. 7. Distribution of $\tau_{z\theta}$ for Case 3, $\beta = 0{\cdot}5$ and $\mu_0 = 0{\cdot}02$.

The effect of μ_0 on the distribution of interface σ_z for the cases of vertical and horizontal concentrated forces are shown in Figs. 9 and 10 respectively. It is observed that the stronger the layer in which the force is applied, the faster the stress σ_z decays in the unloaded layer. It is also true for the stresses τ_{zr} and $\tau_{z\theta}$. However, this viewpoint is not applicable to σ_r, σ_θ and $\tau_{r\theta}$ since discontinuities are allowed at the interface, and the rigidity of the lower layer plays a dominant role in this case. The error introduced by assuming both layers to be homogeneous, i.e. $\mu_0 = 1$, may be large and give a completely wrong picture of the stress distributions in the actual system.

For the case of a vertical line force acting on the surface of a two-layer elastic half plane, good agreements are found between the results obtained by the proposed solution and those given by Lemcoe's solution[12], as shown in Fig. 11. In Westmann's solution[8] for a two-layer half space subjected to a surface shearing force uniformly distributed over a circular

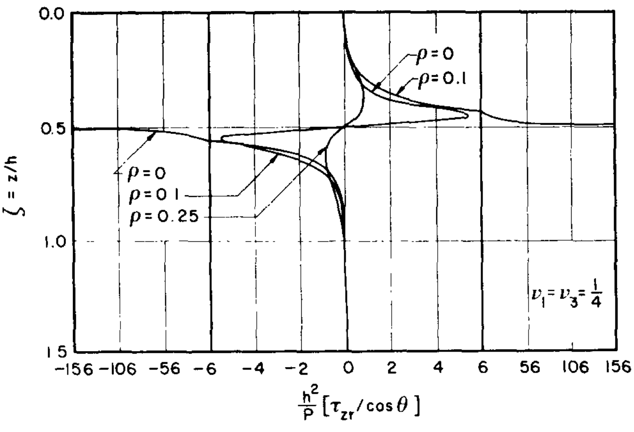

Fig. 8. Distribution of τ_{zr} for Case 3, $\beta = 0{\cdot}5$ and $\mu_0 = 0{\cdot}02$.

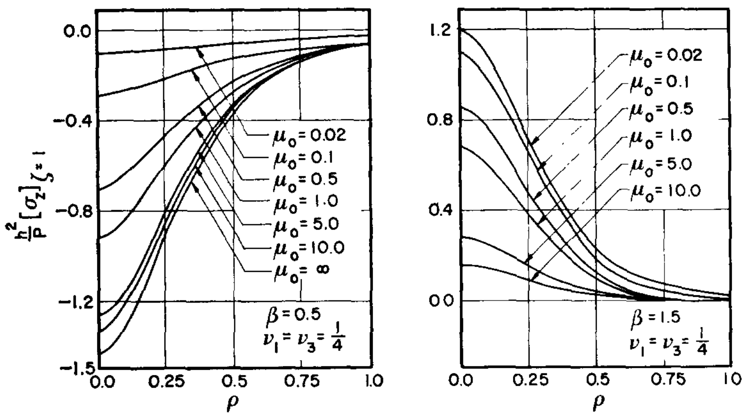

Fig. 9. Effect of μ_0 on interface σ_z due to vertical force.

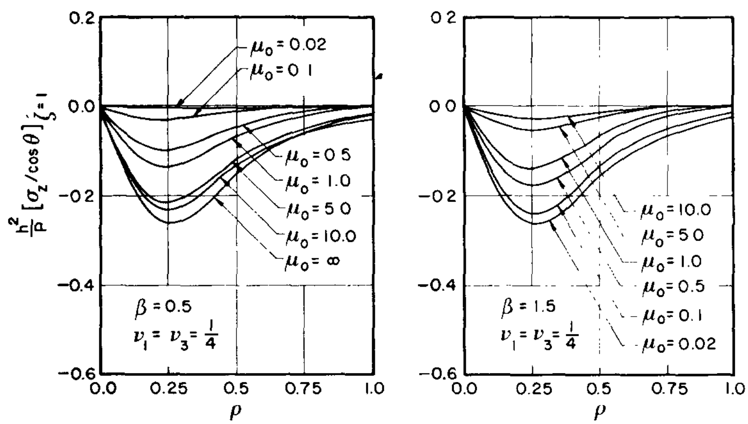

Fig. 10. Effect of μ_0 on interface σ_z due to horizontal force.

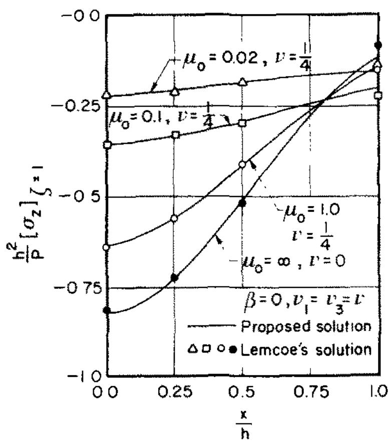

Fig. 11. Comparison of interface σ_z in a half plane problem with Lemcoe's solution.

area, the same common denominators $D(\alpha)$ and $H(\alpha)$ occur in the solution integrals. With the proposed approximations of $1/D(\alpha)$ and $1/H(\alpha)$ for the case where $\mu_0 = 0{\cdot}1$ and $\nu = 1/4$, the computed interface shearing stress τ_{zr} at $\rho = 0$ agrees with Westmann's solution within 0·5 per cent.

7. CONCLUSIONS

The accuracy of the proposed approximations of $1/D(\alpha)$ and $1/H(\alpha)$ has been verified by the results of the proposed solution, which show that the slight violation of the boundary and continuity conditions due to these approximations is not significant for practical purposes. The comparisons of $1/D(\alpha)$ and $1/H(\alpha)$ with their approximations as shown in Fig. 2 are satisfactory even for the worst case. The approximations are asymptotic to unity as $\alpha \to \infty$ and assume exact values at $\alpha = 0$. The latter guarantees that the integration of the traction on a horizontal plane is equal to zero when the plane is above the point of application of the force P, and equal to P when the plane lies below.

The proposed closed form solution yields good results without laborious work required by numerical integration technique[8, 9, 12]. The identification of the singular part of the solution assures good precision in the vicinity of the point of application of the force.

Although the proposed solution for the two-layer half space is approximate, it yields, as limiting cases, the exact solution of Mindlin and Melan as well as of the half space fixed on the surface, the latter a limiting case of Rongved's solution.

The numerical results which show the effect of μ_0 on the stress distributions at a distance away from the force are of practical interest in the analysis and design of foundation systems involving two layers of distinct properties, such as pavement on grade and the anchors of guy wire and suspension cable in stratified soil.

The solution for the case of a force arbitrarily distributed over an area may be obtained by an appropriate superposition of the solution for a force with intensity q_0 uniformly distributed over a small area of radius r_0. The solution of the latter case can be obtained by means of the Hankel transform $\bar{q}(\eta)$ of the applied traction $q(r)$ equal to $q_0 J_1(r_0\eta)/(\pi r_0 \eta)$.

The extension of the proposed solution to a multi-layer system, while laborious, presents no fundamental difficulty.

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Абстракт — На основе полумембранной теории оболочек определяется новая дискретно-сплошная аналитическая модель для призматических оболочек с шестиугольным поперечным сечением. Выводятся окончательно результаты для напряжений в виде несложных выражений.

Проверяется предлагаемая модель с точки зрения анализа и результатов конечного элемента. Оценивается точность во всех исследованных случаях, которая исключительно надлежаща. Поэтому оказывается, что этой метод дает простый, но мощный инструмент, в области проектировки подсборных трубопроводов ядерных реакторов.

APPENDIX A—STANDARD INTEGRALS IN CLOSED FORMS

The infinite integrals involving Bessel functions of the form

$$G[m, n, p] = \int_0^\infty \alpha^n e^{-p\alpha} J_m(\rho\alpha)\, d\alpha \tag{46}$$

are given in closed forms as follows. For $p > 0$,

$$G[0, 0, p] = \frac{1}{R} \tag{A1}$$

$$G[0, 1, p] = \frac{p}{R^3} \tag{A2}$$

$$G[0, 2, p] = -\frac{1}{R^3}\left(1 - \frac{3p^2}{R^2}\right) \tag{A3}$$

$$G[0, 3, p] = -\frac{3p}{R^5}\left(3 - \frac{5p^2}{R^2}\right) \tag{A4}$$

$$G[0, 4, p] = \frac{3}{R^5}\left(3 - \frac{30p^2}{R^2} + \frac{35p^4}{R^4}\right) \tag{A5}$$

$$G[1, 0, p] = \frac{\rho}{R(R+p)} \tag{A6}$$

$$G[1, 1, p] = \frac{\rho}{R^3} \tag{A7}$$

$$G[1, 2, p] = \frac{3\rho p}{R^5} \tag{A8}$$

$$G[1, 3, p] = -\frac{3\rho}{R^5}\left(1 - \frac{P^2}{R^2}\right) \tag{A9}$$

$$G[1, 4, p] = \frac{-15\rho p}{R^7}\left(3 - \frac{7p^2}{R^2}\right) \tag{A10}$$

$$G[2, 0, p] = \frac{\rho^2}{R(R+p)^2} \tag{A11}$$

$$G[2, 1, p] = \frac{\rho^2}{R^2(R+p)^2}\left(2 + \frac{p}{R}\right) \tag{A12}$$

$$G[2, 2, p] = \frac{3\rho^2}{R^5} \tag{A13}$$

$$G[2, 3, p] = \frac{15\rho^2 p}{R^7} \tag{A14}$$

$$G[2, 4, p] = \frac{15\rho^2}{R^7}\left(6 - \frac{7\rho^2}{R^2}\right) \tag{A15}$$

where

$$R = (\rho^2 + p^2)^{1/2}. \tag{A16}$$

For plane problems, the infinite integrals of the forms

$$G_s[n, p] = \int_0^\infty \alpha^{(n-1)} e^{-p\alpha} \sin \xi\alpha \, d\alpha \tag{100}$$

$$G_c[n, p] = \int_0^\infty \alpha^{(n-1)} e^{-p\alpha} \cos \xi\alpha \, d\alpha \tag{101}$$

are given in closed forms as follows. For $n \geqslant 0$ and $p > 0$,

$$G_s[0, p] = \tan^{-1}(\xi/p) + \infty \tag{A17}$$

$$G_s[1, p] = \frac{\xi}{R_1^2} \tag{A18}$$

$$G_s[2, p] = \frac{2\xi p}{R_1^4} \tag{A19}$$

$$G_s[3,p] = \frac{2\xi}{R_1^4}\left(3 - \frac{4\xi^2}{R_1^2}\right) \tag{A20}$$

$$G_s[4,p] = \frac{6\xi p}{R_1^6}\left(4 - \frac{8\xi^2}{R_1^2}\right) \tag{A21}$$

$$G_c[0,p] = -\ln(R_1) \tag{A22}$$

$$G_c[1,p] = \frac{p}{R_1^2} \tag{A23}$$

$$G_c[2,p] = \frac{1}{R_1^2}\left(\frac{2p^2}{R_1^2} - 1\right) \tag{A24}$$

$$G_c[3,p] = \frac{2p}{R_1^4}\left(\frac{4p^2}{R_1^2} - 3\right) \tag{A25}$$

$$G_c[4,p] = \frac{6}{R_1^4}\left[\frac{8p^2}{R_1^2}\left(\frac{p^2}{R_1^2} - 1\right) + 1\right] \tag{A26}$$

where

$$R_1 = (\xi^2 + p^2)^{1/2}. \tag{A27}$$

APPENDIX B—APPROXIMATIONS OF $1/D(\alpha)$ AND $1/H(\alpha)$.

The approximations of $1/D(\alpha)$ and $1/H(\alpha)$ for $\nu_1 = \nu_3 = 1/4$ and various values of μ_0 are given as follows. For $\mu_0 = 0{\cdot}02$,

$$1/D(\alpha) \simeq 1 + 18{\cdot}74e^{-2\alpha} + 112{\cdot}3e^{-5\alpha} - 144{\cdot}2e^{-8\alpha} - 44{\cdot}98e^{-11\alpha} + 139{\cdot}4e^{-14\alpha} - 106{\cdot}2e^{-17\alpha} - 452{\cdot}5e^{-20\alpha} + 1060{\cdot}0e^{-23\alpha}$$

$$1/H(\alpha) \simeq 1 + 1{\cdot}783e^{-2\alpha} - 18{\cdot}24e^{-5\alpha} + 123{\cdot}5e^{-8\alpha} - 255{\cdot}9e^{-11\alpha} + 173{\cdot}3e^{-14\alpha}$$

for $\mu_0 = 0{\cdot}1$,

$$1/D(\alpha) \simeq 1 + 16{\cdot}43e^{-2\alpha} - 45{\cdot}49e^{-4\alpha} + 109{\cdot}4e^{-6\alpha} - 94{\cdot}07e^{-8\alpha} - 40{\cdot}42e^{-10\alpha} + 81{\cdot}13e^{-12\alpha}$$

$$1/H(\alpha) \simeq 1 + 0{\cdot}7114e^{-2\alpha} + 3{\cdot}057e^{-4\alpha} - 15{\cdot}4e^{-6\alpha} + 45{\cdot}12e^{-8\alpha} - 55{\cdot}49e^{-10\alpha} + 26{\cdot}5e^{-12\alpha}$$

for $\mu_0 = 0{\cdot}5$,

$$1/D(\alpha) \simeq 1 + 3{\cdot}474e^{-2\alpha} - 10{\cdot}66e^{-4\alpha} + 14{\cdot}84e^{-6\alpha} - 6{\cdot}433e^{-8\alpha}$$

$$1/H(\alpha) \simeq 1 + 0{\cdot}3328e^{-2\alpha} + 0{\cdot}1162e^{-4\alpha} + 0{\cdot}0207e^{-6\alpha} + 0{\cdot}0302e^{-8\alpha}$$

for $\mu_0 = 5{\cdot}0$,

$$1/D(\alpha) \simeq 1 - 0{\cdot}9117e^{-2\alpha} + 0{\cdot}3684e^{-4\alpha} + 0{\cdot}01217e^{-6\alpha} - 0{\cdot}1266e^{-8\alpha}$$

$$1/H(\alpha) \simeq 1 - 0{\cdot}6647e^{-2\alpha} + 0{\cdot}4219e^{-4\alpha} - 0{\cdot}212e^{-6\alpha} + 0{\cdot}0549e^{-8\alpha}$$

for $\mu_0 = 10{\cdot}0$,

$$1/D(\alpha) \simeq 1 - 1{\cdot}081e^{-\alpha} + 0{\cdot}5355e^{-2\alpha} - 0{\cdot}1211e^{-3\alpha} - 0{\cdot}0526e^{-4\alpha}$$

$$1/H(\alpha) \simeq 1 - 0{\cdot}8139e^{-2\alpha} + 0{\cdot}6189e^{-4\alpha} - 0{\cdot}3534e^{-6\alpha} + 0{\cdot}0984e^{-8\alpha}$$

and, for $\mu_0 = \infty$,

$$1/D(\alpha) \simeq 1 - 1{\cdot}269e^{-\alpha} + 0{\cdot}6996e^{-2\alpha} - 0{\cdot}1938e^{-3\alpha} - 0{\cdot}0135e^{-4\alpha}$$

$$1/H(\alpha) \simeq 1 - 0{\cdot}9909e^{-2\alpha} + 0{\cdot}8915e^{-4\alpha} - 0{\cdot}5695e^{-6\alpha} + 0{\cdot}1689e^{-8\alpha}.$$

The approximations of $1/D(\alpha)$ and $1/H(\alpha)$ for other values of Poisson's ratio are presented in [16].